Supersonic Compressive Ramp Without Laminar Boundary-Layer Separation

George Emanuel
University of Oklahoma, Norman, Oklahoma

Similar, laminar, boundary-layer theory for a two-dimensional or axisymmetric body is extended by a derivation that yields explicit equations for the wall shape. Isentropic edge conditions are assumed, which results in a differential equation for the pressure gradient parameter β. The solution of this equation, when β is constant, parametrically yields the wall shape with the edge Mach number as the parameter. A two-dimensional wall shape is determined when the freestream is supersonic. For a compressive turn, the boundary layer does not separate if β is not too negative. In this case, the magnitude of β depends on the ratio of wall temperature to the freestream stagnation temperature. For this application a criterion is provided for the validity of the isentropic edge assumption. A transformation is given for axisymmmetric body shapes.

Nomenclature

a = defined by Eq. (21d)
B = defined by Eq. (13)
f = nondimensional longitudinal velocity
F = defined by Eq. (8)
g = stagnation temperature ratio
g,h = x,y wall coordinates in terms of arc length
j = 0 for two-dimensional body, 1 for axisymmetric body
M = Mach number
n = coordinate normal to the wall
p = pressure
r = radius of axisymmetric body
r_c = radius of curvature of body in the streamwise direction at station 1
R = gas constant
s = arc length along the wall
T = temperature
u = longitudinal velocity component
x = coordinate along the axis of the body
X = defined by Eq. (5a)
y = coordinate transverse to the axis of the body
β = pressure gradient parameter
β_0 = constant β value
γ = ratio of specific heats
η = similarity variable
θ = wall slope
κ = defined by Eq. (19)
μ = viscosity
μ = Mach angle
ν = Prandtl-Meyer function
ξ = transformed wall arc length
ρ = density
ϕ = defined by Eq. (16a)

Subscripts

ax = axisymmetric
c = location where shock wave forms
e = boundary-layer edge
sp = separation condition
t = stagnation
w = wall
1 = start of β_0 solution
2d = two-dimensional

Superscripts

$(\ )'$ = differentiation with respect to η or s
$(\tilde{\ })$ = dimensional coordinate
$(\hat{\ })$ = dummy integration variable

Introduction

SIMILAR, laminar, boundary-layer theory for steady, low-speed flow is a well-developed subject.[1,2] The theory for a compressible flow approximately dates from Refs. 3 and 4, with reviews in Refs. 1, 5, and 6. For a compressible boundary layer, similarity requires that the edge speed u_e be given by[5]

$$u_e \sim (T_e/T_{te})^{1/2}\xi^{\beta/2} \tag{1}$$

where T is the temperature, ξ the transformed arc length along the wall, and β the (constant) pressure gradient parameter. Except for special shapes, such as a cone at zero angle of attack, Eq. (1) does not easily yield the wall's shape, especially in a supersonic flow.[6]

One objective of this work is to make explicit the wall shape for a given β value. This result is then applied to a supersonic flow over a two-dimensional or axisymmetric body where the wall shape is directly computed. This procedure is of value when β is negative and the wall generates an adverse pressure gradient. Normally this type of flow has a lambda shock system with boundary-layer interaction and separation. However, for the shapes generated here, only a single shock occurs; and the wall can be contoured so that the shock initially forms outside of the boundary layer. Shock-wave/boundary-layer interaction is thereby minimized. Within the context of the theory, boundary-layer separation is avoided, even for a moderate turn angle, provided the length of the wall can be accommodated.

The next section provides a formulation of compressible similarity theory with emphasis on simplifying the equation for β. This extension holds for subsonic or supersonic flow with either a favorable or an adverse pressure gradient. The following section applies the formulation to a two-dimensional or axisymmetric ramp in a supersonic flow. Results are summarized in the final section.

Analysis

Formulation

Based on the assumptions given in Table 1, the well-known boundary-layer equations[5,7] are obtained,

$$f''' + ff'' + \beta(g - f'^2) = 0 \tag{2a}$$

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*Professor of Aerospace, Mechanical and Nuclear Engineering.

$$g''+fg'=0 \tag{2b}$$

The prime indicates differentiation with respect to η, and

$$f'=u/u_e \tag{3a}$$

$$g=T_t/T_{te} \tag{3b}$$

$$\beta=\frac{2\xi}{u_e}\frac{du_e}{d\xi}\frac{T_{te}}{T_e} \tag{3c}$$

$$\xi=\int_0^{\tilde{s}}(\rho\mu u)_e\tilde{r}^{2j}d\hat{s} \tag{3d}$$

$$\eta=\frac{(\rho u)_e\tilde{r}^j}{(2\xi)^{1/2}}\int_0^{\tilde{n}}\frac{\rho}{\rho_e}d\hat{n} \tag{3e}$$

Equations (2) are subject to the customary boundary conditions,

$$f(0)=f'(0)=0,\quad g(0)=g_w=T_w/T_{te},\quad f'(\infty)=g(\infty)=1 \tag{4}$$

Transformation of β

Assumption 3 in Table 1 allows ξ and β to be transformed by means of the relations,

$$X=1+(\gamma-1)M_e^2/2 \tag{5a}$$

$$u_e=(\gamma RT_{te})^{1/2}M_eX^{-1/2} \tag{5b}$$

$$T_e=T_{te}X^{-1} \tag{5c}$$

$$p_e=p_{te}X^{-\gamma/(\gamma-1)} \tag{5d}$$

$$(\rho\mu)_e=(\mu/RT)_wp_e \tag{5e}$$

Table 1 Boundary-layer assumptions

1)	Steady two-dimenisonal or axisymmetric flow
2)	Similar, laminar, boundary-layer theory without wall injection
3)	Isentropic boundary-layer edge conditions
4)	Calorically and thermally perfect gas
5)	Prandtl number equal to unity and viscosity proportional to the temperature

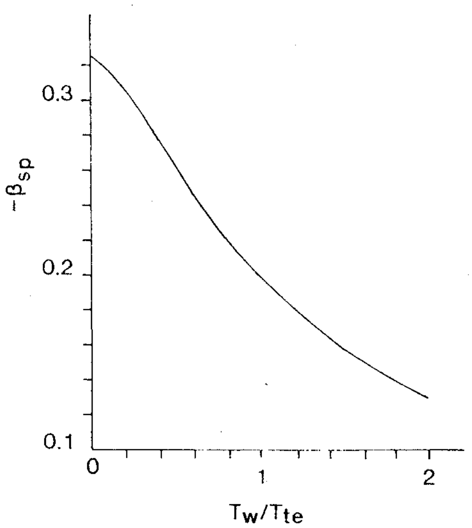

Fig. 1 Separation value of the pressure gradient parameter vs temperature ratio.[7]

Thus we obtain

$$\xi=\left(\frac{\mu}{RT}\right)_w(\gamma RT_{te})^{1/2}p_{te}\tilde{s}_1\tilde{r}_1^{2j}\int_0^s r^{2j}F\mathrm{d}\hat{s} \tag{6}$$

$$\beta=\frac{2}{M_e}\frac{\mathrm{d}M_e}{\mathrm{d}s}\frac{\int_0^s r^{2j}F\mathrm{d}\hat{s}}{r^{2j}F} \tag{7}$$

where F and the shape variables are given by

$$F(M_e)=M_eX^{-(3\gamma-1)/[2(\gamma-1)]} \tag{8}$$

$$s=\tilde{s}/\tilde{s}_1 \tag{9a}$$

$$r=\tilde{r}/\tilde{r}_1 \tag{9b}$$

Note that Eq. (1) is not utilized and that the constant β similarity requirement is imposed later. The location of the start of the boundary layer is $s=0$, while after $s=1$, β is to be held constant at a value given by

$$\beta=\beta_0 \tag{10}$$

Separation is predicted when $f''(0)=0$. The corresponding value for β, denoted as β_{sp}, is a function only of T_w/T_{te}, as can be seen from Eqs. (2) and (4). Accurate values for β_{sp} are provided in Ref. 7 and are graphically shown in Fig. 1. In an adverse pressure gradient, laminar separation is thus avoided by using a wall shape with β_0 bounded by

$$\beta_{sp}<\beta_0 \tag{11}$$

A minimum wall length for the compression, without separation, is then obtained by keeping β_0 close to β_{sp}.

In the integral in Eq. (7), r and F are functions of the integration variable $\hat{s}$. A differential equation for β is directly obtained by differentiating Eq. (7) to obtain

$$\frac{\mathrm{d}\beta}{\mathrm{d}s}=\left[-\frac{1}{M_e}\frac{\mathrm{d}M_e}{\mathrm{d}s}+\left(\frac{\mathrm{d}M_e}{\mathrm{d}s}\right)^{-1}\frac{\mathrm{d}^2M_e}{\mathrm{d}s^2}-\frac{\mathrm{d}M_e}{\mathrm{d}s}\frac{\mathrm{d}\ell nF}{\mathrm{d}M_e}\right]\times\frac{2}{M_e}\frac{\mathrm{d}M_e}{\mathrm{d}s}\frac{\int_0^s r^{2j}F\mathrm{d}\hat{s}}{r^{2j}F}+\frac{2}{M_e}\frac{\mathrm{d}M_e}{\mathrm{d}s}-\frac{2j\beta}{r}\frac{\mathrm{d}r}{\mathrm{d}M_e}\frac{\mathrm{d}M_e}{\mathrm{d}s} \tag{12a}$$

Since

$$\frac{\mathrm{d}\ell nF}{\mathrm{d}M_e}=\frac{1-\gamma M_e^2}{M_eX}$$

Eq. (12a) can be written as

$$\frac{1}{\beta}\frac{\mathrm{d}\beta}{\mathrm{d}s}=\left(\frac{\mathrm{d}M_e}{\mathrm{d}s}\right)^{-1}\frac{\mathrm{d}^2M_e}{\mathrm{d}s^2}+\left(\frac{B}{M_e}-\frac{2j}{r}\frac{\mathrm{d}r}{\mathrm{d}M_e}\right)\frac{\mathrm{d}M_e}{\mathrm{d}s} \tag{12b}$$

where

$$B=\left(\frac{\gamma+1}{2}M_e^2-2\right)X^{-1}+\frac{2}{\beta} \tag{13}$$

For local similarity computations Eq. (12b) is usually a more convenient form for β than is Eqs. (3c) or (7).

Equation for $M_e(s)$

Applying Eq. (10) to Eq. (12b) yields

$$\frac{\mathrm{d}^2M_e}{\mathrm{d}s^2}+\left(\frac{B}{M_e}-\frac{2j}{r}\frac{\mathrm{d}r}{\mathrm{d}M_e}\right)\left(\frac{\mathrm{d}M_e}{\mathrm{d}s}\right)^2=0 \tag{14}$$

with $\beta=\beta_0$ in B. This differential equation is used with the initial conditions

$$M_e(1)=M_1 \qquad \frac{dM_e}{ds}(1)=M_1' \tag{15a,b}$$

to provide a wall shape that satisfies Eq. (10). The derivative M_1' is determined by Eq. (7) with β replaced by β_0. If the β_0 ramp is preceeded by a flat plate where β is zero, then Eq. (7) yields

$$M_1'=\tfrac{1}{2}M_1\beta_0 \tag{15c}$$

By replacing s with $\bar{s}$ in the integral in Eq. (7), one obtains the same result for a curved ramp with a sharp leading edge.

When the ramp is preceeded by a wall with a different β value, a region of rapid β change occurs in the vicinity of $s=1$. For a supersonic flow, β can change discontinuously since dM_e/ds can be discontinuous on Mach lines. A rapid change in β, however, results in nonsimilar boundary-layer effects, which are not dealt with here. Therefore, the preferred intepretation for Eq. (15c) is that of a curved ramp with a sharp leading edge.

By setting,

$$\phi=\frac{dM_e}{ds} \qquad \frac{d\phi}{dM_e}=\frac{1}{\phi}\frac{d^2M_e}{ds^2} \tag{16a,b}$$

Eq. (14) becomes

$$\frac{d\phi}{dM_e}+\left(\frac{B}{M_e}-\frac{2j}{r}\frac{dr}{dM_e}\right)\phi=0 \tag{17}$$

which integrates to

$$\frac{dM_e}{ds}=\kappa r^{2j}M_e^{(1-2/\beta_0)}F(M_e) \tag{18}$$

where

$$\kappa=\frac{M_1'}{M_1^{(1-2/\beta_0)}F(M_1)} \tag{19}$$

A second integration yields

$$\kappa\int_1^s r^{2j}d\hat{s}=\int_{M_1}^{M_e}\frac{d\hat{M}}{\hat{M}^{(1-2/\beta_0)}F(\hat{M})} \tag{20}$$

Equation (20) is not needed in the subsequent development.

Wall Shape

The wall shape from $s=1$ to a downstream point can be written in terms of the arc length as (see Fig. 2)

$$x=g(s) \tag{21a}$$

$$y=h(s) \tag{21b}$$

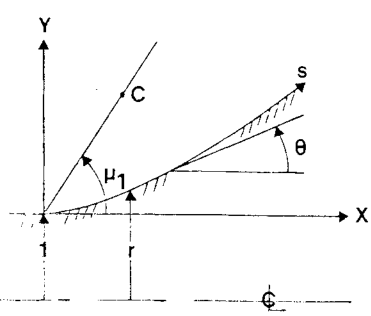

Fig. 2 Nondimensional coordinate schematic, where r is present only for axisymmetric flow.

$$r=1+ay \tag{21c}$$

$$a=\bar{s}_1/\bar{r}_1 \tag{21d}$$

where x and y are nondimensionalized by $\bar{s}_1$. One can readily show that

$$dx=\cos\theta ds \tag{22a}$$

$$dy=\sin\theta ds \tag{22b}$$

In the two-dimensional case, this result plus Eq. (18) yields

$$x_{2d}=\frac{1}{\kappa}\int_{M_1}^{M_e}\frac{\cos\theta}{\hat{M}^{(1-2/\beta_0)}F(\hat{M})}d\hat{M} \tag{23a}$$

$$y_{2d}=\frac{1}{\kappa}\int_{M_1}^{M_e}\frac{\sin\theta}{\hat{M}^{(1-2/\beta_0)}F(\hat{M})}d\hat{M} \tag{23b}$$

In the axisymmetric case, we have

$$y_{ax}+ay_{ax}^2+\frac{1}{3}a^2y_{ax}^3=\frac{1}{\kappa}\int_{M_1}^{M_e}\frac{\sin\theta}{\hat{M}^{(1-2/\beta_0)}F(\hat{M})}d\hat{M} \tag{24a}$$

$$x_{ax}=\frac{1}{\kappa}\int_{M_1}^{M_e}\frac{\cos\theta}{(1+ay_{ax})^2\hat{M}^{(1-2/\beta_0)}F(\hat{M})}d\hat{M} \tag{24b}$$

In the above, θ is a function of $\hat{M}$, which is discussed later. Equations (23) and (24) parametrically yield the wall shape, with M_e as the parameter. These equations hold for a subsonic or a supersonic freestream and for a favorable (positive β_0) or an adverse pressure gradient. For all cases, x is positive, since the integrands in Eqs. (23a) and (24b) are positive, and

$$M_e\le M_1 \text{ when } \kappa<0$$

$$M_e\ge M_1 \text{ when } \kappa>0$$

Recall that the sign of κ is given by M_1', which, in turn, is given by β_0. On the other hand, y increases with x when 1) $M_1>1$ and $\beta_0<0$, or 2) $M_1<1$ and $\beta_0>0$. Otherwise it decreases.

For later use, the radius of curvature r_c of the wall in the streamwise direction at $s=1$ is needed. This is given by

$$r_c=\left(\frac{ds}{d\theta}\right)_1=\left[M_1'\left(\frac{d\theta}{dM_e}\right)_1\right]^{-1} \tag{25}$$

Supersonic Ramp

Two-Dimensional Body

For supersonic flow over a convex or concave ramp, the angle θ is given by

$$\theta=\nu(M_1)-\nu(M_e) \tag{26}$$

where ν is the Prandtl-Meyer function. The concave ramp is of interest since it generates a shock wave whose interaction

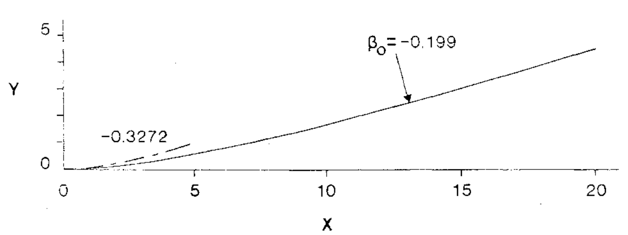

Fig. 3 Wall shape when $M_1=2$, M_1' is given by Eq. (15c), $\gamma=1.4$, and Mach number at the end of the ramp is 1.4.

with the boundary layer can lead to separation. However, separation can be avoided if the wall is contoured in accord with Eq. (11), even for a moderate turn angle.

Implicit in this claim is the assumption of isentropic edge conditions. This assumption presumes that the Mach lines first coalesce when outside of the boundary layer. The coordinate system shown in Fig. 2 is used, where point c is the location where the shock forms. The coordinates of this point are[8]

$$x_c = (M_l^2 - 1)^{1/2} y_c \tag{27a}$$

$$y_c = \frac{2r_c}{(\gamma+1)} \frac{M_l^2 - 1}{M_l^4} \tag{27b}$$

When θ is given by Eq. (26), Eq. (25) yields for r_c

$$r_c = -\frac{X_l}{(M_l^2 - 1)^{1/2}} \frac{M_l}{M_l'} \tag{28}$$

where x_c, y_c, and r_c are nondimensionalized by $\bar{s}_l$. If point c falls outside of the boundary layer, then the isentropic assumption is warrented. The principal adjustment for moving point c away from the wall is to decrease M_l' by increasing β_0.

When a laminar boundary layer experiences an adverse pressure gradient, it may become transitional. Separation should be less likely and the suggested approach, therefore, is conservative.

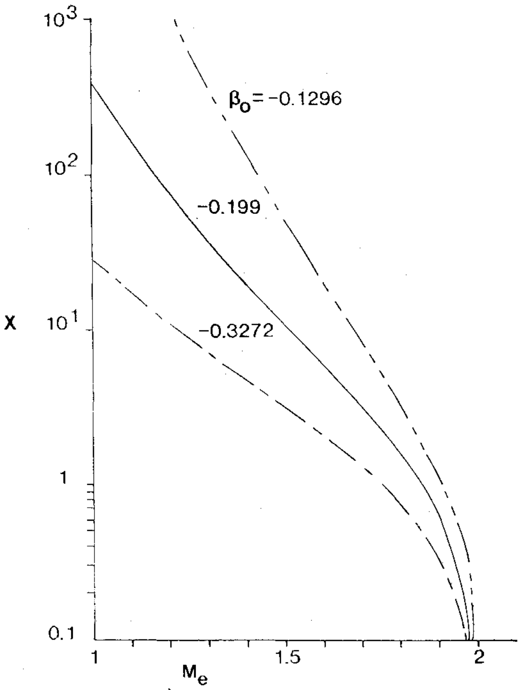

Fig. 4a x vs M_e, when $M_l = 2$, M_l' is given by Eq. (15c), and $\gamma = 1.4$.

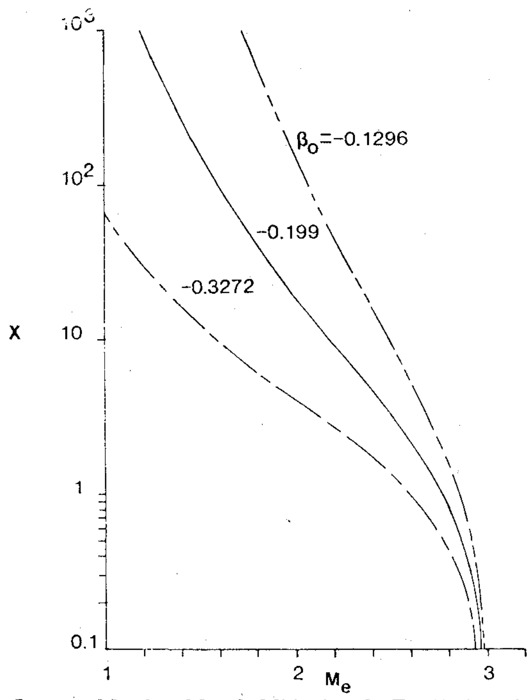

Fig. 5a x vs M_e when $M_l = 3$, M_l' is given by Eq. (15c), and $\gamma = 1.4$.

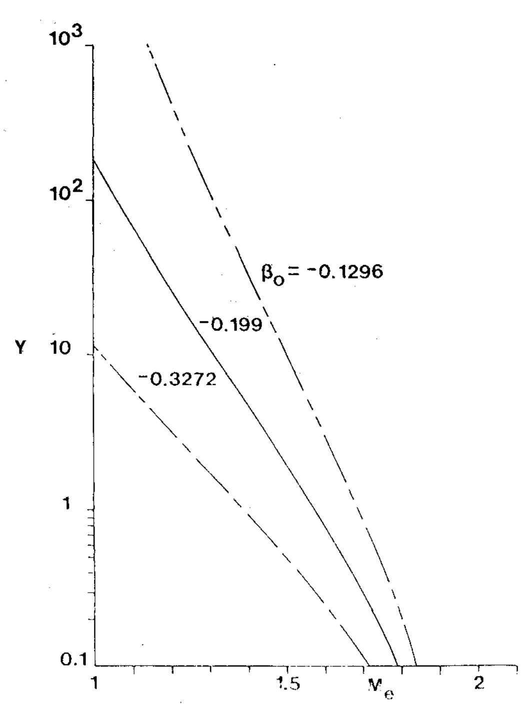

Fig. 4b y vs M_e when $M_l = 2$, M_l' is given by Eq. (15c), and $\gamma = 1.4$.

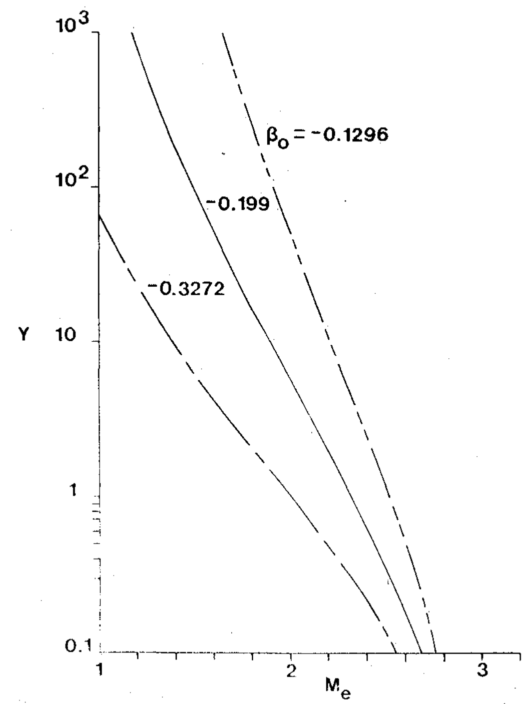

Fig. 5b y vs M_e when $M_l = 3$, M_l' is given by Eq. (15c), and $\gamma = 1.4$.

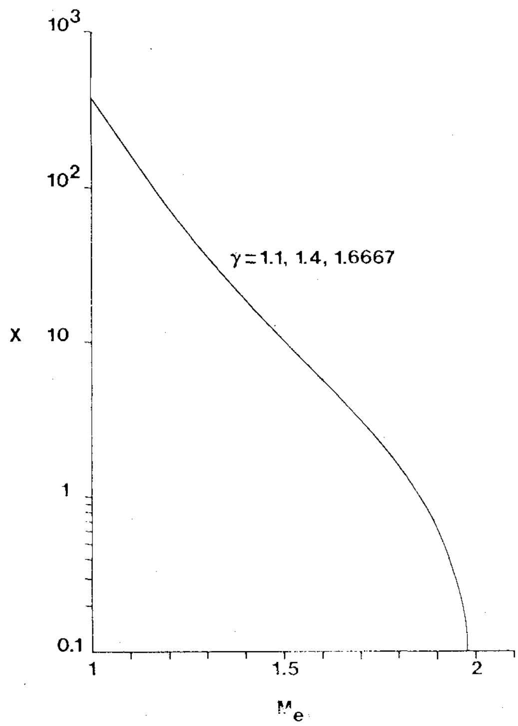

Fig. 6a x **vs** M_e **when** $M_I=2$, M_I' **is given by Eq. (15c),** $\beta_0=-0.199$, **and** γ **is 1.1, 1.4, or 1.6667.**

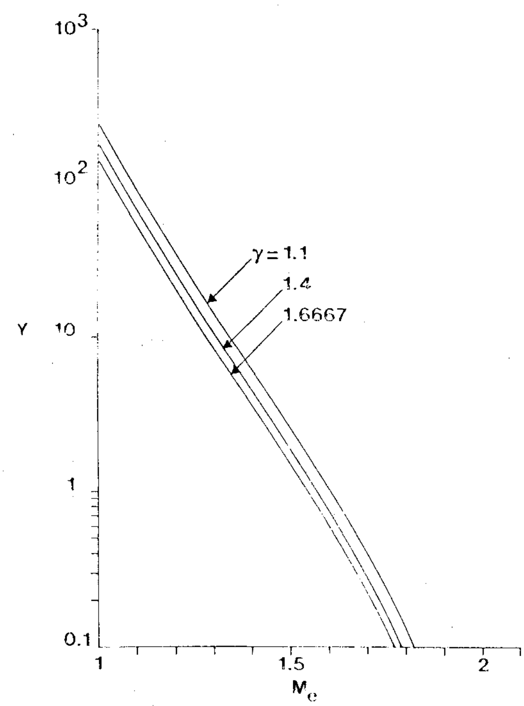

Fig. 6b y **vs** M_e **when** $M_I=2$, M_I' **is given by Eq. (15c),** $\beta_0=-0.199$, **and** γ **is 1.1, 1.4, or 1.6667.**

Results

Figures 3-6 shows results for a two-dimensional ramp that utilizes Eq. (15c). For Figs. 3-5, the ratio of specific heats is 1.4 and the β_0 values correspond to boundary-layer separation when T_w/T_{te} is 0, 1, or 2, in accord with Fig. 1. (An adiabatic wall has unity for the temperature ratio.)

Figure 3, which does not include a $\beta_0=-0.1296$ curve, shows the nondimensional wall shape when $M_I=2$ and the Mach number at the downstream end is 1.4. The adiabatic wall is four times longer than when the wall is cold and demonstrates the sensitivity of a compressive wall shape to β_0. The slow increase in Fig. 3 of wall slope is typical of compressive walls when M_e does not change significantly. For large changes in M_e, the downstream wall slope becomes appreciable, as is evident in Fig. 5.

Figures 4 and 5 show x and y vs M_e when M_I is two and three, respectively. While M_I is fixed, the downstream supersonic Mach number at the termination of the ramp is arbitrary. These figures show the long wall length required for a significant change in M_e. In this regard, a cold wall is clearly desirable. This rapid increase in wall length is largely due to the factor $\hat{M}^{-2/\beta_0}$ in Eqs. (23). For supersonic flow, this factor varies rapidly when β_0 has a small magnitude.

Figure 6 shows the lack of sensitivity to γ. In fact, the different x curves could not be separated to the scale of Fig. 6a.

Axisymmetric Body

For a sufficiently small a, Eq. (26) provides a first estimate for the wall slope when the body is axisymmetric. In this case, the right sides of Eqs. (23b) and (24a) are identical, and Eq. (24a) can be written as

$$3ay_{\rm ax}+3a^2y_{\rm ax}^2+a^3y_{\rm ax}^3=3ay_{\rm 2d} \tag{29a}$$

or

$$y_{\rm ax}=(1/a)\left[(1+3ay_{\rm 2d})^{1/3}-1\right] \tag{29b}$$

The x coordinate is then given by Eq. (24b) with $(1+ay_{\rm ax})^2$ replaced by $(1+3ay_{\rm 2d})^{2/3}$ and with the upper limit in Eq. (23b) changed to $\hat{M}$ for the $y_{\rm 2d}$ factor. The axisymmetric case is thus expedited by first performing the two-dimensional computation, which is followed by a single quadrature for $x_{\rm ax}$. Equation (29) presume M_I, β_0, and κ are the same for the two flows. If κ is not the same, then replace $y_{\rm 2d}$ by $(\kappa_{\rm 2d}/\kappa_{\rm ax})y_{\rm 2d}$ in the above discussion.

Summary and Discussion

Similar, laminar, boundary-layer theory for a two-dimensional or axisymmetric body is extended by a derivation that yields explicit equations for the wall shape. The analysis is based on assumptions contained in Table 1. A key result is Eq. (12b), which is a differential equation for the pressure gradient parameter as a function of wall arc length. This equation may prove useful in analyses where local similarity is assumed.

Results are given for a flat plate followed by a two-dimensional compressive ramp with a supersonic freestream. Wall shape is sensitive to the chosen value for β and the wall is typically quite long. Nevertheless, the theory my prove useful in the design of the supersonic section of a chemical laser diffuser, where the cross section is rectangular. Because of the low density and high temperature, the Reynolds number for this flow is small; and, at least at early run times, cold wall conditions prevail.

The closest experimental results the author is aware of utilize an ogive-cylinder body followed by a 10 deg half-angle cone.[9] The freestream Mach number is 8 and apparently separation upstream of the cone occurred for all test conditions. The extent of separation, however, diminished sharply when the wall temperature is reduced. (Also see Ref. 10.)

Acknowledgments

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